Point Charges

Liénard-Wiechert Potentials

Retarded potentials, $V(r,t)$ and $A(r,t)$ of a point charge 'q' that is moving on a specified trajectory

$$w(t) \equiv \text{position of } q \text{ at time } 't' \quad \text{--- (1)}$$

The retarded time is determined implicitly by the eq^n

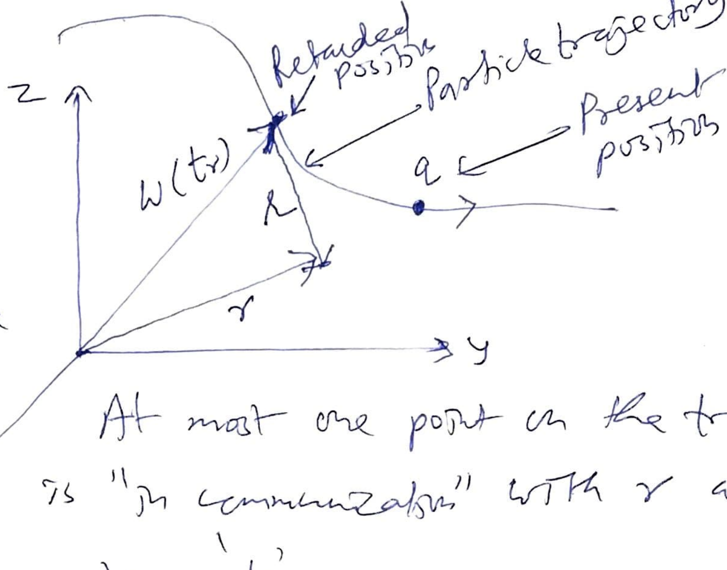
$$|r - w(t_r)| = c(t - t_r), \quad \text{--- (2)}$$

Left side → distance the "new" must travel, and $(t - t_r)$ → time it takes to make the trip.

$w(t_r)$ → retarded position of the charge

$\vec{\mathscr{r}}$ → vector from the retarded position to the field point r:

$$\vec{\mathscr{r}} = r - w(t_r) \quad \text{--- (3)}$$



At most one point on the trajectory is "in communication" with r at any particular time, 't'.

For suppose there were two such points, with retarded times t_1 and t_2:

$$\mathscr{r}_1 = c(t - t_1) \text{ and } \mathscr{r}_2 = c(t - t_2)$$

Then $\mathscr{r}_1 - \mathscr{r}_2 = c(t_2 - t_1)$ → so the average velocity of the particle in the direction of r would have to be 'c' — no counting of velocity in other direction.

No charged particle can travel at speed of light
⟹ only one retarded point contribute to the potentials at any given moment.

formula $V(r,t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r', t_r)}{\mathscr{r}} d\tau'$ — (4)

might suggest that → retarded potential of a point charge is simply

$$\frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}}$$

→ same as in the static case, $\mathscr{r}$ → distance to the retarded position of the charge

But this is wrong ✗

Reason ?

It is true that for a point ~~charge~~ source the denominator $\mathscr{r}$ comes out of the integral but what remains

$$\int \rho(r', t_r)\, d\tau' \quad \text{— (5)}$$

is not equal to the charge of the particle.

To calculate total charge of a configuration → must integrate ρ over the entire distribution at one instant of time, but here the retardation $t_r = t - \frac{\mathscr{r}}{c}$ → evaluate ρ at different times for different parts of the configuration.

If source is moving → distorted picture of the total charge.

✗ This problem would disappear for point charges?

→ in terms of charge & current densities

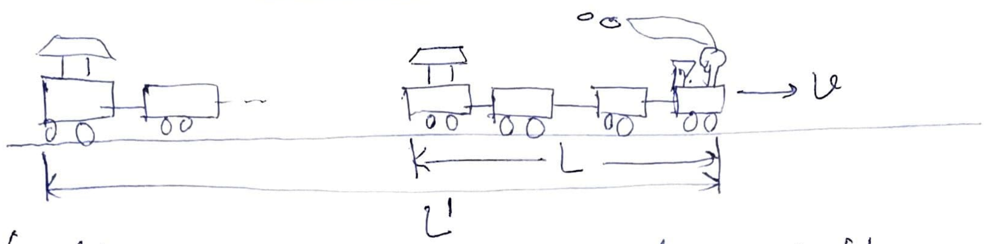
Maxwell's electrodynamics → a point charge must be ~~regar~~ regarded as the limit of an extended charge, when the size goes to zero

For an extended particle → retardation is eq^n (5) throws in a factor $(1 - \hat{\boldsymbol{\imath}}\cdot\frac{\mathbf{v}}{c})^{-1}$, where $\mathbf{v}$ is the velocity of the charge at the retarded time:

$$\int \rho(\mathbf{r}', t_r)\, d\tau' = \frac{q}{1 - \hat{\boldsymbol{\imath}}\cdot\frac{\mathbf{v}}{c}} \quad \text{--- (6)}$$

Proof:- This is purely "geometrical effect"

Train coming towards you → looks a little longer than it really is



Light you receive from the caboose left earlier than the light you receive simultaneously from the engine, and that earlier time train was farther away

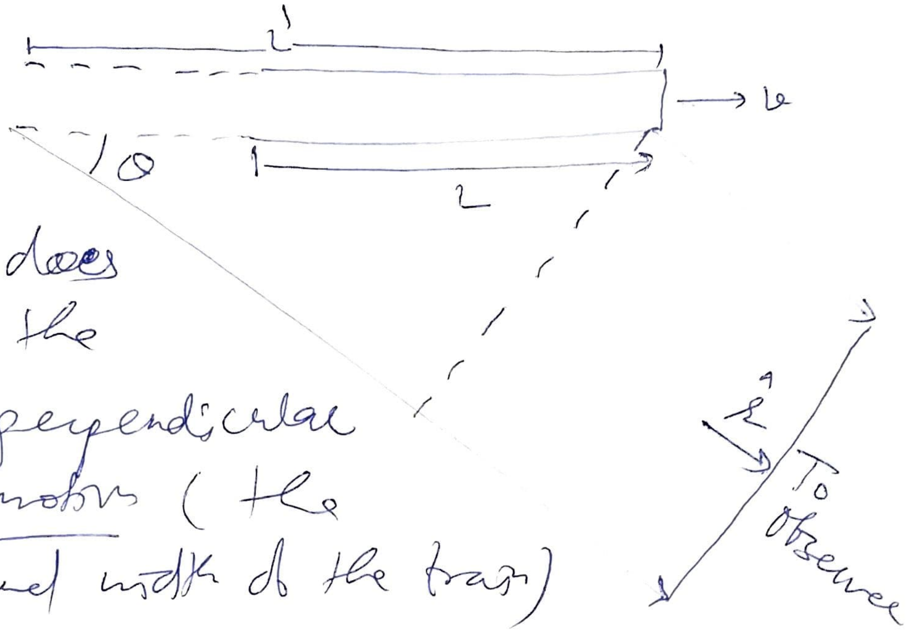
In the interval it takes light from the caboose to travel the extra distance L', the train itself moves a distance $L' - L$:

$$\frac{L'}{c} = \frac{L' - L}{v} \quad \text{or} \quad L' = \frac{L}{1 - \frac{v}{c}}$$

So approaching train appears longer, by a factor $(1 - \frac{v}{c})^{-1}$. By contrast a train going away from you looks shorter by a factor $(1 + \frac{v}{c})^{-1}$.

In general, if the train's velocity makes an angle θ with your line of sight, the extra distance light from the caboose must travel is $L'\cos\theta$.
In the time $\frac{L'\cos\theta}{c}$, the train moves a distance $(L' - L)$:

$$\frac{L'\cos\theta}{c} = \frac{L'-L}{v} \quad \text{or} \quad L' = \frac{L}{1 - v\cos\theta/c}$$



This effect does not distort the dimensions perpendicular to the motion (the height and width of the train)

The apparent volume τ' of the train, is related to the actual volume τ by

$$\tau' = \frac{\tau}{1 - \hat{r}\cdot\frac{v}{c}}, \quad \text{— (7)}$$

$\hat{r} \rightarrow$ unit vector from the train to observer

It follows then

$$\boxed{V(r,t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(rc - \vec{r}\cdot v)}} \quad \text{— (8)}$$

$v \rightarrow$ velocity of charge at the retarded time

$\vec{\mathscr{r}} \rightarrow$ vector from the retarded position to the field point r.

Current density of a rigid object $\rightarrow \rho v$

$$A(r,t) = \frac{\mu_0}{4\pi}\int \frac{\rho(r', t_r)\, v(t_r)}{\mathscr{r}}\, d\tau'$$

$$= \frac{\mu_0}{4\pi}\frac{v}{\mathscr{r}}\int \rho(r', t_r)\, d\tau'$$

$$\boxed{A(r,t) = \frac{\mu_0}{4\pi}\frac{qcv}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot v)} = \frac{v}{c^2}V(r,t)}$$

— (9)

Eqⁿ (8), (9) $\rightarrow$ Liénard-Wiechert potentials for a moving point charge

The Fields of a Moving Point Charge

Potential of a point charge moving with constant velocity

Say the particle passes through the origin at time $t=0$, so that

$$w(t) = vt$$

retarded time

$$|r - vt_r| = c(t - t_r)$$

$$\Rightarrow r^2 - 2r\cdot v t_r + v^2 t_r^2 = c^2(t^2 - 2tt_r + t_r^2)$$

Solving for t_r by the quadratic eqⁿ

$$t_r = \frac{(c^2 t - r\cdot v) \pm \sqrt{(c^2t - r\cdot v)^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

To fix the sign, consider the limit $v = 0$

$$t_r = t \pm \frac{r}{c}$$

In this case charge is at rest at the origin and the retarded time should be $\left(t - \frac{r}{c}\right)$; evidently we want minus sign

$$\Rightarrow \quad \mathscr{r} = c\,(t - t_r) \text{ and } \hat{\mathscr{r}} = \frac{r - v t_r}{c\,(t - t_r)}$$

So $$\mathscr{r}\left(1 - \hat{\mathscr{r}}\cdot\frac{v}{c}\right) = c\,(t - t_r)\left[1 - \frac{v}{c}\cdot\frac{r - v t_r}{c\,(t - t_r)}\right]$$

$$= c\,(t - t_r) - \frac{v\cdot r}{c} - \frac{v^2}{c} t_r$$

$$= \frac{1}{c}\left[(c^2 t - r\cdot v) - (c^2 - v^2) t_r\right]$$

$$= \frac{1}{c}\sqrt{(c^2 t - r\cdot v)^2 + (c^2 - v^2)(r^2 - c^2 t^2)}$$

$$\Rightarrow \quad V(r,t) = \frac{1}{4\pi\varepsilon_0}\frac{q c}{\sqrt{(c^2 t - r\cdot v)^2 + (c^2 - v^2)(r^2 - c^2 t^2)}} \quad \text{--- (2)}$$

and

$$A(r,t) = \frac{\mu_0}{4\pi}\frac{q c v}{\sqrt{(c^2 t - r\cdot v)^2 + (c^2 - v^2)(r^2 - c^2 t^2)}} \quad \text{--- (3)}$$